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EACH REGULAR NUMBER STRUCTURE IS BIREGULAR

BY

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ABSTRACT

Roughly speaking we show that for certain number structures $\mathfrak{A}$, $\mathfrak{B}$ with $\mathfrak{B} \subseteq \mathfrak{A}$, if $\mathfrak{B}$ is bounded above in $\mathfrak{A}$ then $\mathfrak{B}$ is bounded below in $\mathfrak{A}$.

0. Introduction

In this note we answer a question posed by Hirschfeld in [2]. All the results of [2] that we require are contained in [3] and since [3] is more accessible than [2] we refer to [3] rather than [2].

Hirschfeld's results are concerned with the class $\mathscr{E}_N$ of structures which are e.c. for full number theory N. He considers several subclasses of $\mathscr{E}_N$, in particular he introduces (in [3, 10.1 and 10.14(ii)]) the subclasses $\mathscr{R}_N$, $\mathscr{B}_N$ of regular and biregular structures. He shows (in [3, 10.16]) that $\mathscr{B}_N \subseteq \mathscr{R}_N$ and asks (in [3, 12.14(ii)]) whether $\mathscr{B}_N = \mathscr{R}_N$. The main result of this note (i.e. Theorem 9) is a proof of this equality. We also strengthen a remark of [3, p. 160, last paragraph] and use this to give a simple proof of [3, 11.9].

This note is written in the style of [1], [4] and so we assume a slight familiarity with these papers. In particular we use the notation, terminology, etc. of [1], [4], sometimes without explanation.

Notice that since [1], [4] are concerned with a wider context than [3] we will require certain simple minded generalizations of some of the results of [3].

1. Required lemmas

Let N be full first order number theory formalized in some suitable language, let P be peano number theory formalized in the same language, and let B be the theory axiomatized by $P \cap \forall_2$. (For us a theory is a deductively closed set of

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sentences.) We are concerned with certain models of B. (In [3] only models of $N \cap \forall_2$ are considered.)

First we require two formal properties of B.

Let us say a formula θ is a Δ-formula if all the quantifiers of θ are bounded. The following result is essentially the solution of Hilbert's 10th problem.

LEMMA 1. *Let θ be a Δ-formula. Then there is an $\exists_1$-formula ϕ and an $\forall_1$-formula ψ, each having the same free variables as θ, such that* $B \vdash \theta \leftrightarrow \phi$ *and* $B \vdash \theta \leftrightarrow \psi$.

This lemma shows that, for many purposes, Δ-formulas can be treated as quantifier-free formulas.

We will require the following normal form for certain $\exists_1$-formulas.

LEMMA 2. *There is a Δ-formula $T(u, v, w, x)$ such that for each $\exists_1$-formula $\theta(v, w)$ there is some $n \in \omega$ with*

$$B \vdash \theta(v, w) \leftrightarrow D(v, w, n)$$

where $D(v, w, x)$ is $(\exists u) T(u, v, w, x)$.

Let $\mathscr{E}$ be the class of models of B which are e.c., so $\mathfrak{A} \in \mathscr{E}$ if and only if $\mathfrak{A} \models B$ and for each model $\mathfrak{B}$ of B,

$$\mathfrak{A} \subseteq \mathfrak{B},\ \mathfrak{A} \equiv_1 \mathfrak{B} \Rightarrow \mathfrak{A} <_1 \mathfrak{B}.$$

We easily check that

$$\mathfrak{B} <_1 \mathfrak{A} \in \mathscr{E} \Rightarrow \mathfrak{B} \in \mathscr{E}$$

and that for all $\mathfrak{A}, \mathfrak{B} \in \mathscr{E}$

$$\mathfrak{B} <_1 \mathfrak{A} \Rightarrow \mathfrak{B} <_2 \mathfrak{A}.$$

(Notice that the class studied in [3] is $\mathscr{E}_N = \mathscr{E} \cap \mathrm{Md}(N \cap \forall_2)$.)

For each model $\mathfrak{A}$ of B and subset X of A (the carrier set of $\mathfrak{A}$) we put

$$K(\mathfrak{A}, X) = \cap \{\mathfrak{B} : \mathfrak{B} <_1 \mathfrak{A}, X \subseteq B\}.$$

In particular, $K(\mathfrak{A}, \varnothing)$ is the core $K(\mathfrak{A})$ of $\mathfrak{A}$ considered in [4, §5].

Using [4, 1.3] and the above remarks we obtain the following.

LEMMA 3. *Let $\mathfrak{A} \in \mathscr{E}$ and $X \subseteq A$. Then $K(\mathfrak{A}, X) \in \mathscr{E}$ and $K(\mathfrak{A}, X) <_2 \mathfrak{A}$.*

We are concerned with $K(\mathfrak{A}, X)$ for finite X only. Notice that

$$K(\mathfrak{A}, \{a_0, \cdots, a_r\}) = K(\mathfrak{A}, \{a\})$$

where $a = 2^{a_0}3^{a_1}\cdots p_r^{a_r}$, so we may assume X is a singleton. We write

$$K(\mathfrak{A}, a) \quad \text{for} \quad K(\mathfrak{A}, \{a\}).$$

There is a useful characterization of the elements of $K(\mathfrak{A}, a)$. This next lemma is proved in the same way as [4, 5.6].

LEMMA 4. *Let $\mathfrak{A} \models$ B and let a be an element of $\mathfrak{A}$. For each element b of $\mathfrak{A}$ the following are equivalent*:

i) *b is an element of $K(\mathfrak{A}, a)$;*

ii) *there is some $n \in \omega$ such that*

$$\mathfrak{A} \models D(b, a, n)$$

and

$$\mathfrak{A} \models (\forall v_1, v_2)[D(v_1, a, n) \wedge D(v_2, a, n) \rightarrow v_1 = v_2].$$

Next we need to know that the standard part of certain models of B is definable. Part (i) of the next lemma is just [4, 2.4] and part (ii) is an easy generalization of [3, 8.29].

LEMMA 5.

i) *There is an $\exists_2$-formula $I(v)$ such that for each $\mathfrak{A} \in \mathscr{E}$ and element a of $\mathfrak{A}$,*

$$\mathfrak{A} \models I(a) \Leftrightarrow a \in \omega.$$

ii) *There is an $\forall_1$-formula $J(v, w)$ such that for each $\mathfrak{A} \in \mathscr{E}$ and elements a, p of $\mathfrak{A}$, if $K(\mathfrak{A}) < p$ then*

$$\mathfrak{A} \models J(a, p) \Leftrightarrow a \in \omega.$$

Finally we need an overspill principle. The following lemma is essentially the result of [3, p. 154].

LEMMA 6 ($\exists_1$-overspill). *Let $\psi(y, \mathbf{x})$ be an $\exists_1$-formula (where y and the finite sequence $\mathbf{x}$ are the only free variables of ψ). Let $\mathfrak{A} \in \mathscr{E}$ be such that $K(\mathfrak{A})$ is bounded above in $\mathfrak{A}$ and let $\mathbf{a}$ be a sequence of elements of $\mathfrak{A}$ (exactly matching $\mathbf{x}$). If, for each $n \in \omega$, $\mathfrak{A} \models \psi(n, \mathbf{a})$ then there is some infinite element q of $\mathfrak{A}$ such that $\mathfrak{A} \models \psi(q, \mathbf{a})$.*

2. The results

The following definition extends [3, 10.1].

DEFINITION. *A* model $\mathfrak{A}$ of B is regular if $\mathfrak{A} \in \mathscr{E}$ and for each element a of $\mathfrak{A}$, $K(\mathfrak{A}, a)$ is bounded above in $\mathfrak{A}$. We let $\mathscr{R}$ be the class of regular models of B.

Notice that the regular structures of [3] are just the members of

$$\mathcal{R}_N = \mathcal{R} \cap \mathcal{E}_N = \mathcal{R} \cap \mathrm{Md}(\mathrm{N} \cap \forall_2).$$

Our first theorem strengthens the remark in [3, p. 160, last paragraph].

THEOREM 7. *For each $\exists_2$-formula $\phi(\boldsymbol{w}, \boldsymbol{x})$ there is a Δ-formula $\psi(\boldsymbol{w}, \boldsymbol{x}, y)$ such that the following holds. For each $\mathfrak{A} \in \mathcal{R}$, sequence $\boldsymbol{a}$ of elements of $\mathfrak{A}$ (matching $\boldsymbol{x}$), element b of $\mathfrak{A}$ such that $K(\mathfrak{A}, \boldsymbol{a}) < b$, and sequence $\boldsymbol{n}$ of elements of ω (matching $\boldsymbol{w}$),*

$$\mathfrak{A} \models \phi(\boldsymbol{n}, \boldsymbol{a}) \Leftrightarrow \mathfrak{A} \models \psi(\boldsymbol{n}, \boldsymbol{a}, b).$$

PROOF. There is some quantifier-free formula $\theta(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}, \boldsymbol{x})$ such that ϕ is the formula $(\exists \boldsymbol{u})(\forall \boldsymbol{v})\theta(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}, \boldsymbol{x})$. Let ψ be the formula

$$(\exists \boldsymbol{u} < y)(\forall \boldsymbol{v} < y)\theta(u, v, \boldsymbol{w}, \boldsymbol{x}).$$

The required result now follows using Lemma 3.

This theorem can be used to give a simple proof of [3, 11.9].

For each $\mathfrak{A} \in \mathcal{R}$ let $\Delta(\mathfrak{A})$ be the set of number theoretic relations $R(\boldsymbol{w})$ such that for some Δ-formula $\theta(\boldsymbol{w}, \boldsymbol{x})$ and sequence $\boldsymbol{a}$ of elements of $\mathfrak{A}$,

$$R(\boldsymbol{n}) \Leftrightarrow \mathfrak{A} \models \theta(\boldsymbol{n}, \boldsymbol{a})$$

(where, of course, $\boldsymbol{n}$ is a sequence of elements of ω).

THEOREM 8. *Let $\mathfrak{A} \in \mathcal{R}$. Then $\Delta(\mathfrak{A})$ is closed under arithmetical definability.*

PROOF. Clearly $\Delta(\mathfrak{A})$ is closed under the propositional connectives, so it is sufficient to show that $\Delta(\mathfrak{A})$ is closed under existential quantification.

Let $R(v, \boldsymbol{w}) \in \Delta(\mathfrak{A})$ and let $\theta(v, \boldsymbol{w}, \boldsymbol{x})$ be a Δ-formula and $\boldsymbol{a}$ a sequence of elements of $\mathfrak{A}$ such that

$$R(m, \boldsymbol{n}) \Leftrightarrow \mathfrak{A} \models \theta(m, \boldsymbol{n}, \boldsymbol{a}).$$

Then

$$(\exists v) R(v, \boldsymbol{n}) \Leftrightarrow \mathfrak{A} \models (\exists v)[I(v) \wedge \theta(v, \boldsymbol{n}, \boldsymbol{a})]$$

so that, since the right hand formula is B-equivalent to an $\exists_2$-formula, the required result follows by Theorem 7.

Finally we come to the main result of this note.

Given two models $\mathfrak{A}, \mathfrak{B}$, of B such that $\mathfrak{B} \subseteq \mathfrak{A}$, we say $\mathfrak{B}$ is bounded below in $\mathfrak{A}$ if there is an *infinite* element d of $\mathfrak{A}$ such that for each *infinite* element e of $\mathfrak{B}$, $d < e$.

THEOREM 9. *Let $\mathfrak{A} \in \mathscr{R}$ and let a be an element of $\mathfrak{A}$. Then $K(\mathfrak{A}, a)$ is bounded below in $\mathfrak{A}$.*

PROOF. Since $\mathfrak{A} \in \mathscr{R}$ there is some element p of $\mathfrak{A}$ such that $K(\mathfrak{A}, a) < p$. Consider the set $X \subseteq \omega$ defined by

$$n \in X \Leftrightarrow \mathfrak{A} \models (\exists v)[\neg J(v, p) \wedge D(v, a, n)].$$

By Theorem 7 there is some Δ-formula $\theta(x, w_1, w_2, y)$ and element b of $\mathfrak{A}$ such that

$$n \in X \Leftrightarrow \mathfrak{A} \models \theta(n, a, p, b).$$

Let $\phi(y, z, a, p, b)$ be the Δ-formula (with parameters a, p, b)

$$(\forall x < y)[\theta(x, a, p, b) \rightarrow (\exists v < p)(\exists u < p)[T(u, v, a, x) \wedge z < v]]$$

and let $\psi(y, a, p, b)$ be the $\exists_1$-formula

$$(\exists x)(\neg J(z, p) \wedge \phi(y, z, a, p, b)].$$

For each $n \in X$ let c_n be an element of $\mathfrak{A}$ such that

$$\mathfrak{A} \models \neg J(c_n, p), \quad \mathfrak{A} \models D(c_n, a, n).$$

Notice that Lemma 5(ii) shows that c_n is infinite.

Consider any $m \in \omega$ and let d be an infinite element of $\mathfrak{A}$ such that

$$d < \min\{c_n : n \in X, n < m\}.$$

If $K(\mathfrak{A}, a)$ is bounded below in $\mathfrak{A}$ by d then we are done. If not then there is some infinite element c of $K(\mathfrak{A}, a)$ such that $c < d$, and so for each $n \in X$, $n < m$

$$\mathfrak{A} \models (\exists v)[D(v, a, n) \wedge c < v].$$

But $K(\mathfrak{A}, a) <_2 \mathfrak{A}$ so for each $n \in X$, $n < m$,

$$K(\mathfrak{A}, a) \models (\exists v)[D(v, a, n) \wedge c < v].$$

Thus, for each $n \in X$, $n < m$,

$$\mathfrak{A} \models (\exists v < p)(\exists u < p)[T(u, v, a, n) \wedge c < v]$$

so that $\mathfrak{A} \models \phi(m, c, a, p, b)$. Hence, since c is an infinite element of $K(\mathfrak{A}, a)$,

$$\mathfrak{A} \models \psi(m, a, p, b).$$

Now ψ is an $\exists_1$-formula so that Lemma 6 gives us an infinite element q of $\mathfrak{A}$ such that

$$\mathfrak{A} \models \psi(q, a, p, b).$$

In particular there is some infinite element d of $\mathfrak{A}$ such that

$$(\alpha) \qquad n \in X \Rightarrow \mathfrak{A} \models (\exists v)[D(v, a, n) \wedge d < v].$$

We show that $K(\mathfrak{A}, a)$ is bounded below in $\mathfrak{A}$ by d.

Let e be any infinite element of $K(\mathfrak{A}, a)$. So

$$(\beta) \qquad \mathfrak{A} \models \neg J(e, p)$$

and, by Lemma 4, there is some $n \in \omega$ such that

$$(\gamma) \qquad \mathfrak{A} \models D(e, a, n)$$

$$(\delta) \qquad \mathfrak{A} \models (\forall v_1, v_2)[D(v_1, a, n) \wedge D(v_2, a, n) \rightarrow v_1 = v_2].$$

From (β), (γ) we see that $n \in X$ and so (α), (δ) give $d < e$, as required.

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